(Q.) The electric potential of some configuration is given by the expression $V(\vec{r}) = A\dfrac{e^{-\lambda r}}{r}$,

where A and λ are constants. Find the electric field $\vec{E}(\vec{r})$, the charge density $\rho(r)$ and total charge Q.

Soln. Since we know that electric field $\vec{E}$ is given by the negative gradient of electric potential i.e., $\vec{E} = -\nabla V$

$$\vec{E} = -\frac{\partial}{\partial r}\left[A\frac{e^{-\lambda r}}{r}\right]\hat{r} = -A\left[-\lambda\frac{e^{-\lambda r}}{r} - \frac{1}{r^2}e^{-\lambda r}\right]\hat{r}$$

$$\text{or } \vec{E} = A\frac{e^{-\lambda r}}{r^2}[1+\lambda r]\hat{r}$$

$$\text{or } \boxed{\vec{E} = A e^{-\lambda r}[1+\lambda r]\frac{\hat{r}}{r^2}} \quad \text{--- (1)}$$

Charge density ρ is given by the expression

$$\rho = \epsilon_0 \nabla\cdot\vec{E} = \epsilon_0 A\, \nabla\cdot\left\{e^{-\lambda r}(1+\lambda r)\frac{\hat{r}}{r^2}\right\}$$

$$\text{or } \cancel{\rho = \epsilon_0 A\, \nabla\cdot\left\{e^{-\lambda r}\frac{\hat{r}}{r^2} + \lambda e^{-\lambda r}\frac{\hat{r}}{r}\right\}}$$

$$\text{or } \rho = \epsilon_0 A\left[e^{-\lambda r}(1+\lambda r)\,\nabla\cdot\frac{\hat{r}}{r^2} + \frac{\hat{r}}{r^2}\cdot\nabla\{e^{-\lambda r}(1+\lambda r)\}\right] \quad \text{--- (2)}$$

We use properties and some relations involving Delta function.

$$f(x)\,\delta(x-a) = f(a)\,\delta(x-a) \quad \text{--- (3)}$$

$$\nabla\cdot\left(\frac{\hat{r}}{r^2}\right) = -4\pi\delta^3(\vec{r}) \quad \text{--- (4)}$$

Now eqn (2) can be written as

$$\rho = \epsilon_0 A\left[4\pi e^{-\lambda r}(1+\lambda r)\,\delta^3(\vec{r}) + \frac{\hat{r}}{r^2}\cdot\left\{\hat{r}\frac{\partial}{\partial r}\left(e^{-\lambda r}(1+\lambda r)\right)\right\}\right]$$

In above expression we have used relation (4). and using relation (3) we can write $e^{-\lambda r}(1+\lambda r)\,\delta^3(\vec{r})$

$$= \delta^3(\vec{r})$$

$$\rho = \epsilon_0 A \left[4\pi\delta^3(\vec{r}) + \frac{1}{r^2}\left\{ -\lambda e^{-\lambda r}(\cancel{1}+\lambda r) + \cancel{e^{-\lambda r}\lambda} \right\}\right]$$

or $$\rho = \epsilon_0 A \left[4\pi\delta^3(\vec{r}) + \frac{1}{r^2}\left\{ -r\lambda^2 e^{-\lambda r} \right\}\right]$$

or $$\boxed{\rho = \epsilon_0 A \left[4\pi\delta^3(\vec{r}) - \frac{\lambda^2}{r} e^{-\lambda r}\right]}$$

Next, the total charge Q is given by

$$Q = \int \rho \, dV$$

$$Q = \int \epsilon_0 A \left[4\pi\delta^3(\vec{r}) - \frac{\lambda^2}{r} e^{-\lambda r}\right] dV$$

$$Q = \int \epsilon_0 A \, 4\pi\delta^3(\vec{r}) \, dV - \epsilon_0 A \lambda^2 \int \frac{e^{-\lambda r}}{r} dV$$

or $$Q = 4\pi\epsilon_0 A \int \delta^3(\vec{r}) \, dV - \lambda^2 \epsilon_0 A \int_0^\infty \frac{e^{-\lambda r}}{r} \underbrace{4\pi r^2 dr}$$

$$\int \delta^3(\vec{r}) \, dV = 1$$

$$\int dV = \int_{r}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^2 \sin\theta \, dr \, d\theta \, d\phi = \int_r 4\pi r^2 dr$$

$$Q = 4\pi\epsilon_0 A - \lambda^2 \epsilon_0 A \cdot 4\pi \int_0^\infty r \cdot e^{-\lambda r} dr$$

since $$\int_0^\infty r e^{-\lambda r} dr = \frac{1}{\lambda^2}$$

H.W. Prove it

Hint:

use integration by parts

$$\int_0^\infty r e^{-\lambda r} dr = \frac{-r}{\lambda} e^{-\lambda r} - \int 1 \frac{e^{-\lambda r}}{-\lambda} dr$$

$$= -\frac{1}{\lambda}\left[r e^{-\lambda r} + e^{-\lambda r}\right]_0^\infty$$

Next, obtain the limit of above expression for $r \to \infty$

for lower limit put $r = 0$

Now total charge Q is given by

$$Q = 4\pi\epsilon_0 A - \lambda^2 \epsilon_0 A 4\pi \cdot \frac{1}{\lambda^2}$$

$$Q = 4\pi\epsilon_0 A - 4\pi\epsilon_0 A = 0$$

or $$\boxed{Q = 0}$$